

# IT'S ALL RELATIVE: MEAN FIELD GAME EXTENSIONS OF MERTON'S PROBLEM

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# Introduction

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# Merton's portfolio problem

## Merton (1969)

- Merton (following Samuelson) built a reference model for optimal consumption and investment choices.
- Used Hamilton-Jacobi-Bellman (HJB) equation.
- Various settings with closed-form solutions:
  - CRRA utility function.
  - CARA utility function (raises the question of negative consumption).
  - Finite and infinite horizon.

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  - Finite and infinite horizon.

One of Merton's great successes with the Black-Scholes-Merton formula.



# A vast literature

## Many extensions

- Transaction costs
- Taxes
- Labor income
- Stochastic volatility
- Trading constraints
- Habit formation preferences
- Recursive utility
- Partial information
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## Several mathematical methods

- HJB equation
- Dual / Martingale method

# Our contribution

## Goal

- Introduction of comparison / jealousy / competition in Merton's portfolio problem.
- Showing Jean-Michel Lasry an example of MFG Master equation that could be solved in closed form.

*Remark: MFG of controls  $\rightarrow$  not only characterized by a Master equation.*

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- Showing Jean-Michel Lasry an example of MFG Master equation that could be solved in closed form.

*Remark: MFG of controls  $\rightarrow$  not only characterized by a Master equation.*

## Literature I knew on competition in optimal investment

- Static model: Guéant, Lasry, Lions, Mean Field Games and Applications, Paris-Princeton Lectures on Mathematical Finance 2010.
- Dynamic model: Espinosa, Touzi, Optimal Investment under Relative Performance Concerns, Mathematical Finance, 2015.

## Literature I discovered in August (Discussion with René Carmona on MFG)

- Lacker, Zariphopoulou, Mean field and n-agent games for optimal investment under relative performance criteria. Mathematical Finance, 2017.
- Lacker, Soret, Many-player games of optimal consumption and investment under relative performance criteria. arXiv 28 May 2019.

# Our contribution

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## Our framemork

- Less general:
  - A single common stock (the motivation was a Master equation).
  - Agents differ only by their wealth (original MFG framework).
  - No  $N$ -player game, only MFG.
- More general:
  - CARA case (raises the question of negative consumption).
  - Recursive utility case (Epstein-Zin/Duffie-Epstein/Duffie-Lions).

## The model

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# Merton's portfolio problem

## Original problem

- Stock:  $dS_t = \mu S_t dt + \sigma S_t dW_t$ .
- Wealth:  $dX_t = (r + \theta_t(\mu - r))X_t dt + \theta_t \sigma X_t dW_t - c_t dt$ .
- Optimization problem:

$$\sup_{c, \theta} \mathbb{E} \left[ \int_0^T e^{-\rho t} u(c_t) dt + \epsilon e^{-\rho T} u(X_T) \right].$$

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## HJB equation

$$\begin{aligned} 0 = & \partial_t V - \rho V + r x \partial_x V + \sup_c \{ u(c) - c \partial_x V \} \\ & + \sup_{\theta} \left\{ \theta(\mu - r) x \partial_x V + \frac{1}{2} \theta^2 \sigma^2 x^2 \partial_{xx}^2 V \right\} \end{aligned}$$

Terminal condition:  $V(T, x) = \epsilon u(x)$ .

**CARA case**  $u(x) = -\frac{1}{\gamma}e^{-\gamma x}$

- Ansatz:  $V(t, x) = -\frac{1}{\gamma}e^{-\gamma f(t)x+g(t)}$ .
- Resulting equations:

$$0 = f' + rf - f^2, \quad f(T) = 1.$$

$$0 = g' - fg + f - f \log(f) - \rho - \frac{(\mu - r)^2}{2\sigma^2}, \quad g(T) = \log(\epsilon).$$

- Optimal controls (feedback form):

$$c^*(t, x) = f(t)x - \frac{1}{\gamma}(\log(f(t)) + g(t)).$$

$$M^*(t, x) = \theta^*(t, x)x = \frac{\mu - r}{\gamma\sigma^2 f(t)}.$$

**CRRA case**  $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$

- Ansatz:  $V(t, x) = \frac{1}{1-\gamma} f(t) x^{1-\gamma}$ .
- Resulting equation (Bernoulli):

$$f' = \left( \rho - (1-\gamma) \left( r + \frac{(\mu-r)^2}{2\gamma\sigma^2} \right) \right) f - \gamma f^{-\frac{1-\gamma}{\gamma}}$$

with terminal condition  $f(T) = \epsilon$ .

- Optimal controls (feedback form):

$$\begin{aligned} c^*(t, x) &= f(t)^{-\frac{1}{\gamma}} x. \\ \theta(t, x)^* &= \frac{\mu - r}{\gamma\sigma^2}. \end{aligned}$$

# Introducing competition I – the CARA case

## The CARA case

- A population of agents with the same preferences (wealth is distributed).
- A common stock:  $dS_t = \mu S_t dt + \sigma S_t dW_t$ .
- Wealth:  $dX_t = rX_t dt + (\mu - r)M_t dt + \sigma M_t dW_t - c_t dt$ .
- Optimization problem:

$$\sup_{c, \theta} \mathbb{E} \left[ \int_0^T -e^{-\rho t} \frac{1}{\gamma} e^{-\gamma(c_t - \beta \bar{c}_t)} dt - \epsilon e^{-\rho T} \frac{1}{\gamma} e^{-\gamma(X_T - \beta \bar{X}_T)} \right],$$

where  $\beta \in [0, 1)$  and  $(\bar{c}, \bar{X})$  designates averages in the population.

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where  $\beta \in [0, 1)$  and  $(\bar{c}, \bar{X})$  designates averages in the population.

## MFG approach

- Given dynamics of averages, solve HJB to obtain optimal controls.
- Find dynamics of averages consistent with optimal controls (fixed-point problem).

## Dynamics of the averages

$$\bar{c}_t = \bar{c}(t, \bar{X}_t).$$

$$d\bar{X}_t = \bar{\mu}(t, \bar{X}_t)dt + \bar{\sigma}(t, \bar{X}_t)dW_t - \bar{c}(t, \bar{X}_t)dt.$$

# Towards a solution

## Dynamics of the averages

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## HJB equation

$$\begin{aligned}0 &= \partial_t V - \rho V + rx\partial_x V + \sup_c \left\{ -\frac{1}{\gamma} e^{-\gamma(c - \beta\bar{c}(t, \bar{x}))} - c\partial_x V \right\} \\ &+ \sup_M \left\{ (\mu - r)M\partial_x V + \frac{1}{2}\sigma^2 M^2 \partial_{xx}^2 V + \sigma\bar{\sigma}(t, \bar{x})M\partial_{x\bar{x}}^2 V \right\} \\ &+ (\bar{\mu}(t, \bar{x}) - \bar{c}(t, \bar{x}))\partial_{\bar{x}} V + \frac{1}{2}\bar{\sigma}(t, \bar{x})^2 \partial_{\bar{x}\bar{x}}^2 V\end{aligned}$$

Terminal condition:  $V(T, x, \bar{x}) = -\frac{\epsilon}{\gamma} e^{-\gamma(x - \beta\bar{x})}$ .

# Optimal controls and equilibrium conditions

- Ansatz:  $V(t, x, \bar{x}) = -\frac{1}{\gamma} e^{-\gamma f(t)(x - \beta \bar{x}) + g(t)}$ .
- Optimal controls given averages (feedback form):

$$c^*(t, x, \bar{x}) = f(t)(x - \beta \bar{x}) + \beta \bar{c}(t, \bar{x}) - \frac{1}{\gamma} (\log(f(t)) + g(t)).$$

$$M^*(t, x, \bar{x}) = \frac{\mu - r}{\gamma \sigma^2 f(t)} + \beta \frac{\bar{\sigma}(t, \bar{x})}{\sigma}.$$

- Equilibrium equations:

$$\bar{c}(t, \bar{x}) = f(t)\bar{x} - \frac{1}{(1 - \beta)\gamma} (\log(f(t)) + g(t)).$$

$$\bar{\sigma}(t, \bar{x}) = \frac{\mu - r}{(1 - \beta)\gamma \sigma f(t)}.$$

$$\bar{\mu}(t, \bar{x}) = r\bar{x} + \frac{(\mu - r)^2}{(1 - \beta)\gamma \sigma^2 f(t)}.$$

# Solution

- Resulting equations (exactly the same):

$$0 = f' + rf - f^2, \quad f(T) = 1.$$

$$0 = g' - fg + f - f \log(f) - \rho - \frac{(\mu - r)^2}{2\sigma^2}, \quad g(T) = \log(\epsilon).$$

- Optimal controls (feedback form):

$$c^*(t, x) = f(t)x - \frac{1}{(1 - \beta)\gamma} (\log(f(t)) + g(t)).$$

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Everything works as if the risk aversion  $\gamma$  had been replaced by  $(1 - \beta)\gamma$ : agents take more risk with competition in the CARA case!

# Introducing competition II – the CRRA case

## The CRRA case

- A population of agents with the same preferences (wealth is distributed).
- A common stock:  $dS_t = \mu S_t dt + \sigma S_t dW_t$ .
- Wealth:  $dX_t = (r + \theta_t(\mu - r))X_t dt + \sigma \theta_t X_t dW_t - c_t dt$ .
- Optimization problem:

$$\sup_{c, \theta} \mathbb{E} \left[ \int_0^T e^{-\rho t} \frac{1}{1-\gamma} \left( \frac{c_t}{\bar{c}_t^\beta} \right)^{1-\gamma} dt + \epsilon e^{-\rho T} \frac{1}{1-\gamma} \left( \frac{X_T}{\bar{X}_T^\beta} \right)^{1-\gamma} \right],$$

where  $\beta \in [0, 1]$  and  $(\bar{c}, \bar{X})$  designates averages in the population.

*Remark: the averages are arithmetic averages, not geometric as in Lacker-Zariphopoulou and Lacker-Soret, because agents have the same preferences and invest in the same asset.*

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## Dynamics of the averages

$$\begin{aligned}\bar{c}_t &= \alpha(t)\bar{X}_t. \\ d\bar{X}_t &= \bar{\mu}\bar{X}_t dt + \bar{\sigma}\bar{X}_t dW_t - \alpha(t)\bar{X}_t dt.\end{aligned}$$

# Toward a solution

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## HJB equation

$$\begin{aligned}0 &= \partial_t V - \rho V + r x \partial_x V + \sup_c \left\{ \frac{1}{1-\gamma} \left( \frac{c}{\alpha^\beta \bar{x}^\beta} \right)^{1-\gamma} - c \partial_x V \right\} \\ &+ \sup_\theta \left\{ \theta(\mu - r)x \partial_x V + \frac{1}{2} \theta^2 \sigma^2 x^2 \partial_{xx}^2 V + \theta \sigma \bar{\sigma} x \bar{x} \partial_{x\bar{x}}^2 V \right\} \\ &+ (\bar{\mu} - \alpha(t))\bar{x} \partial_{\bar{x}} V + \frac{1}{2} \bar{\sigma}^2 \bar{x}^2 \partial_{\bar{x}\bar{x}}^2 V\end{aligned}$$

Terminal condition:  $V(T, x, \bar{x}) = \frac{\epsilon}{1-\gamma} \left( \frac{x}{\bar{x}^\beta} \right)^{1-\gamma}$ .

# Optimal controls and equilibrium conditions

- Ansatz:  $V(t, x, \bar{x}) = \frac{1}{1-\gamma} f(t) \left(\frac{x}{\bar{x}^\beta}\right)^{1-\gamma}$ .

- Optimal controls given averages (feedback form):

$$c^*(t, x, \bar{x}) = \alpha(t)^{\beta(1-\frac{1}{\gamma})} f(t)^{-\frac{1}{\gamma}} x.$$
$$\theta^*(t, x, \bar{x}) = \frac{\mu - r}{\gamma\sigma^2} - \beta \frac{1-\gamma}{\gamma} \frac{\bar{\sigma}(t, \bar{x})}{\sigma}.$$

- Equilibrium equations:

$$\alpha(t) = f(t)^{-\frac{1}{\gamma+\beta(1-\gamma)}}.$$
$$\bar{\sigma} = \frac{\mu - r}{(\gamma + \beta(1-\gamma))\sigma}.$$
$$\bar{\mu} = r + \frac{(\mu - r)^2}{(\gamma + \beta(1-\gamma))\sigma^2}.$$

# Solution

- Resulting equation (Bernoulli):

$$f'(t) = \left( \rho - (1 - \gamma)(1 - \beta) \left( r + \frac{(\mu - r)^2}{2(\gamma + \beta(1 - \gamma))\sigma^2} \right) \right) f(t) - (\gamma + \beta(1 - \gamma)) f(t)^{-\frac{(1 - \beta)(1 - \gamma)}{\gamma + \beta(1 - \gamma)}}.$$

with terminal condition  $f(T) = \epsilon$ .

- Optimal controls (feedback form):

$$c^*(t, x) = f(t)^{-\frac{1}{\gamma + \beta(1 - \gamma)}} x.$$
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Everything works as if the risk aversion  $\gamma$  had been replaced by  $\gamma + \beta(1 - \gamma) = \beta + (1 - \beta)\gamma$ : competition makes agents behave more like if they had a log utility function.

# Recursive utility

## Epstein-Zin / Kreps-Porteus

- Recursive utility to disentangle risk aversion  $\gamma$  and intertemporal elasticity of substitution (IES)  $\psi$  (CRRA case:  $\psi = \frac{1}{\gamma}$ ).
- Discrete-time version in the no-competition case:

$$U_t = \left[ \rho c_t^{1-\frac{1}{\psi}} + (1-\rho) \mathbb{E} \left[ U_{t+1}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}$$

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$$U_t = \left[ \rho c_t^{1-\frac{1}{\psi}} + (1-\rho) \mathbb{E} \left[ U_{t+1}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}$$

## Equivalent aggregator

The continuous-time counterpart has been studied by Duffie-Epstein / Duffie-Lions who obtained an equivalent “aggregator”:

$$(c, V) \mapsto \frac{\rho}{1-\frac{1}{\psi}} \frac{c^{1-\frac{1}{\psi}} - ((1-\gamma)V)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}}{((1-\gamma)V)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1}}.$$

# Introducing competition III – the recursive utility case

## Recursive utility formulation

- A population of agents with the same preferences (wealth is distributed).
- A common stock:  $dS_t = \mu S_t dt + \sigma S_t dW_t$ .
- Wealth:  $dX_t = (r + \theta_t(\mu - r))X_t dt + \sigma \theta_t X_t dW_t - c_t dt$ .
- Optimization problem:

$$V_t = \sup_{c, \theta} \mathbb{E} \left[ \int_t^T \frac{\rho}{1 - \frac{1}{\psi}} \frac{\left(\frac{c_s}{\bar{c}_s^\beta}\right)^{1 - \frac{1}{\psi}} - ((1 - \gamma)V_s)^{\frac{1 - \frac{1}{\psi}}{1 - \gamma}}}{((1 - \gamma)V_s)^{\frac{1 - \frac{1}{\psi}}{1 - \gamma} - 1}} ds \right. \\ \left. + \frac{(\rho\epsilon)^{\frac{1 - \gamma}{1 - \frac{1}{\psi}}}}{1 - \gamma} \left( \frac{X_T}{\bar{X}_T^\beta} \right)^{1 - \gamma} \right],$$

where  $\beta \in [0, 1]$  and  $(\bar{c}, \bar{X})$  designates averages in the population.

## Toward a solution

### Dynamics of the averages

$$\begin{aligned}\bar{c}_t &= \alpha(t)\bar{X}_t. \\ d\bar{X}_t &= \bar{\mu}\bar{X}_t dt + \bar{\sigma}\bar{X}_t dW_t - \alpha(t)\bar{X}_t dt.\end{aligned}$$

# Toward a solution

## Dynamics of the averages

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## HJB equation

$$\begin{aligned}0 &= \partial_t V + rx\partial_x V + \sup_c \left\{ \frac{\rho}{1 - \frac{1}{\psi}} \frac{\left(\frac{c}{\alpha(t)^\beta \bar{x}^\beta}\right)^{1 - \frac{1}{\psi}}}{((1 - \gamma)V)^{\frac{1 - \frac{1}{\psi}}{1 - \gamma}} - 1} - c\partial_x V \right\} \\ &- \rho \frac{1 - \gamma}{1 - \frac{1}{\psi}} V + \sup_\theta \left\{ \theta(\mu - r)x\partial_x V + \frac{1}{2}\theta^2\sigma^2 x^2 \partial_{xx}^2 V + \theta\sigma\bar{\sigma}x\bar{x}\partial_{x\bar{x}}^2 V \right\} \\ &+ (\bar{\mu} - \alpha(t))\bar{x}\partial_{\bar{x}} V + \frac{1}{2}\bar{\sigma}^2\bar{x}^2\partial_{\bar{x}\bar{x}}^2 V\end{aligned}$$

$$\text{Terminal condition: } V(T, x, \bar{x}) = \frac{(\rho\epsilon)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}}{1-\gamma} \left(\frac{x}{\bar{x}^\beta}\right)^{1-\gamma}.$$

# Optimal controls and equilibrium conditions

- Ansatz:  $V(t, x, \bar{x}) = \frac{1}{1-\gamma} f(t) \left(\frac{x}{\bar{x}^\beta}\right)^{1-\gamma}$ .

- Optimal controls given averages (feedback form):

$$\begin{aligned}c^*(t, x, \bar{x}) &= \rho^\psi \alpha(t)^{\beta(1-\psi)} f(t)^{\frac{1-\psi}{1-\gamma}} x. \\ \theta^*(t, x, \bar{x}) &= \frac{\mu - r}{\gamma \sigma^2} - \beta \frac{1 - \gamma}{\gamma} \frac{\bar{\sigma}(t, \bar{x})}{\sigma}.\end{aligned}$$

- Equilibrium equations:

$$\begin{aligned}\alpha(t) &= \rho^{\frac{1}{\psi} + \beta(1 - \frac{1}{\psi})} f(t)^{-\frac{1 - \frac{1}{\psi}}{1 - \gamma} \frac{1}{\frac{1}{\psi} + \beta(1 - \frac{1}{\psi})}}. \\ \bar{\sigma} &= \frac{\mu - r}{(\gamma + \beta(1 - \gamma))\sigma}. \\ \bar{\mu} &= r + \frac{(\mu - r)^2}{(\gamma + \beta(1 - \gamma))\sigma^2}.\end{aligned}$$

# Solution

- Resulting equation (Bernoulli):

$$f'(t) = \left( \frac{1-\gamma}{1-\frac{1}{\psi}} \rho - (1-\gamma)(1-\beta) \left( r + \frac{(\mu-r)^2}{2(\gamma+\beta(1-\gamma))\sigma^2} \right) \right) f(t) - \frac{1-\gamma}{1-\frac{1}{\psi}} \left( \frac{1}{\psi} + \beta \left( 1 - \frac{1}{\psi} \right) \right) \rho^{\frac{1}{\psi} + \beta \left( 1 - \frac{1}{\psi} \right)} f(t)^{1 - \frac{1-\frac{1}{\psi}}{1-\gamma} \frac{1}{\psi + \beta \left( 1 - \frac{1}{\psi} \right)}}.$$

with terminal condition  $f(T) = (\rho\epsilon)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}$ .

- Optimal controls (feedback form):

$$c^*(t, x) = \rho^{\frac{1}{\psi} + \beta \left( 1 - \frac{1}{\psi} \right)} f(t)^{-\frac{1-\frac{1}{\psi}}{1-\gamma} \frac{1}{\psi + \beta \left( 1 - \frac{1}{\psi} \right)}} x.$$
$$\theta^*(t, x) = \frac{\mu - r}{(\gamma + \beta(1-\gamma))\sigma^2}.$$

# Solution

## Different writing

If  $\gamma_\beta = \gamma + \beta(1 - \gamma)$  and  $\frac{1}{\psi_\beta} = \frac{1}{\psi} + \beta \left(1 - \frac{1}{\psi}\right)$  then

$$f'(t) = \left( \frac{1 - \gamma_\beta}{1 - \frac{1}{\psi_\beta}} \rho - (1 - \gamma_\beta) \left( r + \frac{(\mu - r)^2}{2\gamma_\beta \sigma^2} \right) \right) f(t) - \frac{1 - \gamma_\beta}{1 - \frac{1}{\psi_\beta}} \frac{1}{\psi_\beta} \rho^{\psi_\beta} f(t)^{1 - \frac{1 - \frac{1}{\psi_\beta}}{1 - \gamma_\beta} \psi_\beta}, \quad f(T) = (\rho \epsilon)^{\frac{1 - \gamma_\beta}{1 - \frac{1}{\psi_\beta}}}.$$

Optimal controls (feedback form):

$$c^*(t, x) = \rho^{\psi_\beta} f(t)^{-\frac{1 - \frac{1}{\psi_\beta}}{1 - \gamma_\beta} \psi_\beta} x, \quad \theta^*(t, x) = \frac{\mu - r}{\gamma_\beta \sigma^2}.$$

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If  $\gamma_\beta = \gamma + \beta(1 - \gamma)$  and  $\frac{1}{\psi_\beta} = \frac{1}{\psi} + \beta \left(1 - \frac{1}{\psi}\right)$  then

$$f'(t) = \left( \frac{1 - \gamma_\beta}{1 - \frac{1}{\psi_\beta}} \rho - (1 - \gamma_\beta) \left( r + \frac{(\mu - r)^2}{2\gamma_\beta \sigma^2} \right) \right) f(t) - \frac{1 - \gamma_\beta}{1 - \frac{1}{\psi_\beta}} \frac{1}{\psi_\beta} \rho^{\psi_\beta} f(t)^{1 - \frac{1 - \frac{1}{\psi_\beta}}{1 - \gamma_\beta}} \psi_\beta, \quad f(T) = (\rho \epsilon)^{\frac{1 - \gamma_\beta}{1 - \frac{1}{\psi_\beta}}}.$$

Optimal controls (feedback form):

$$c^*(t, x) = \rho^{\psi_\beta} f(t)^{-\frac{1 - \frac{1}{\psi_\beta}}{1 - \gamma_\beta}} \psi_\beta x, \quad \theta^*(t, x) = \frac{\mu - r}{\gamma_\beta \sigma^2}.$$

Everything works as if the risk aversion  $\gamma$  and the IES  $\psi$  had been replaced by  $\gamma_\beta$  and  $\psi_\beta$ . Competition makes agents behave more like if they had a log utility function in terms of risk and EIS equal to 1.

## What we found

- Closed-form solution for Merton's problem with competition in the CARA, CRRA, and recursive utility cases.
- Examples of MFG of controls with common noise that could be solved in closed form.

# Conclusion

## What we found

- Closed-form solution for Merton's problem with competition in the CARA, CRRA, and recursive utility cases.
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## What's next

Generalizing Lacker-Soret (with different agent types) in the case of Epstein-Zin recursive utility:

*Olivier Guéant, Competition in Merton's problem: the recursive utility case.*



**Thanks for your attention.  
Questions.**